Title

The Kunze–Stein phenomenon

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The convolution inequality. If $\mathbb{G} = SL(2,\mathbb{R})$ then

$$L^2(\mathbb{G}) * L^p(\mathbb{G}) \subseteq L^2(\mathbb{G}), \qquad p \in [1,2).$$

Proved by Kunze–Stein (1960) as a consequence of the analytic continuation of the unitary principal series as uniformly bounded representations.

Consequences (duality + interpolation) :

$$\|f * g\|_{L^p} \lesssim_p \|f\|_{L^2}\|g\|_{L^2}, \qquad p \in (2,\infty].$$

If $A \subseteq \mathbb{G}$ is a bounded open set then

$$\Big|\{x \in \mathbb{G} : |xA \cap A| \geq \gamma|A|\}\Big| \leq C_\gamma,$$

and

$$|A|^{2-\varepsilon} \lesssim_\varepsilon |A \cdot A|, \qquad \varepsilon \in (0,1].$$

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Iwasawa decomposition

We define $\mathbb{G} = SO(d,1) = SO_e(d,1)$ as the connected Lie group of $(d+1)\times(d+1)$ matrices that leave invariant the form

$$[x,y] = x^0y^0 - x^1y^1 - \ldots - x^dy^d.$$

Clearly, $X \in SO(d,1)$ if and only if

$$^{tr}X \cdot I_{d,1} \cdot X = I_{d,1}, \quad \det X = 1, \quad X_{00} > 0,$$

where $I_{d,1}$ is the diagonal matrix $\mathrm{diag}[-1,1,\ldots,1]$.

Let

$$\mathbb{K} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} : k \in SO(d) \right\}.$$

Let

$$\mathbb{M} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m \end{bmatrix} : m \in SO(d-1) \right\}.$$

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Let

$$\mathbb{A} = \left\{ a_s = \begin{bmatrix} \operatorname{ch} s & \operatorname{sh} s & 0 \\ \operatorname{sh} s & \operatorname{ch} s & 0 \\ 0 & 0 & I_{d-1} \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Let

$$\mathbb{N} = \left\{ n_v = \begin{bmatrix} 1 + |v|^2/2 & -|v|^2/2 & {}^{tr}v \\ |v|^2/2 & 1 - |v|^2/2 & {}^{tr}v \\ v & -v & I_{d-1} \end{bmatrix} : v \in \mathbb{R}^{d-1} \right\}.$$

Iwasawa decomposition :

$$\mathbb{G} = \mathbb{KAN}, \qquad dg = e^{2\rho t} dkdtdv, \qquad \rho = (d-1)/2.$$

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Herz majorizing principle (Herz 1970)

Assume $p \in [1,2]$, $\mathbb{G} = SO(d,1)$, $K : \mathbb{G} \to [0,\infty)$. Then

$$\| * K\|_{L^p \to L^p} = \sup_{\|a\|_{L^p} = \|b\|_{L^{p'}} = 1} \int_{\mathbb{G}} \int_{\mathbb{G}} a(g) K(g' g^{-1}) b(g') \, dg dg'$$

$$= \sup_{\|a\|_{L^p} = \|b\|_{L^{p'}} = 1} \int_{\mathbb{K}\times\mathbb{R}\times\mathbb{N}} \int_{\mathbb{K}\times\mathbb{R}\times\mathbb{N}} e^{2\rho t} e^{2\rho t'}$$

$$a(k a_t n) K(k' a_{t'} n' n^{-1} a_{-t} k^{-1}) b(k' a_{t'} n') \, dk dk' dt dt' dn dn'.$$

We use

$$m a_t = a_t m, \quad m n_v = n_v m, \quad a_t n_v a_{-t} = n_{e^t v},$$

for $m \in \mathbb{M}$, $a_t \in \mathbb{A}$, $n_v \in \mathbb{N}$.

It follows that

$$\| * K\|_{L^p \to L^p} \lesssim \sup_{\|A\|_{L^p(\mathbb{K}/\mathbb{M})} = \|B\|_{L^{p'}(\mathbb{K}/\mathbb{M})} = 1} \int_{\mathbb{K}\times\mathbb{K}} A(k)B(k') \left[\int_{\mathbb{R}\times\mathbb{N}} K(k' a_t n k^{-1}) e^{2\rho t/p}\, dt dn \right] dk dk'.$$

If $p = 2$ and $K$is $\mathbb{K}$-biinvariant ($K(kgk') = K(g)$ for any $k, k' \in \mathbb{K}$ and $g \in \mathbb{G}$) it follows that

$$\| * K\|_{L^2 \to L^2} \lesssim \int_{\mathbb{G}} K(g)\Phi_0(g)\, dg.$$

This implies the Kunze–Stein convolution inequality

$$\| * K\|_{L^2 \to L^2} \lesssim_q \|K\|_{L^q(\mathbb{G})}, \qquad q \in [1, 2)$$

in this case, since $\|\Phi_0\|_{L^r(\mathbb{G})} \lesssim_r 1$, $r \in (2, \infty]$.

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Returning to the general case and setting $p = 2$, we need to prove that

$$\int_{\mathbb{K}\times\mathbb{K}} A(k)B(k')\Big[\int_{\mathbb{R}\times\mathbb{N}} K(k'a_t nk^{-1})e^{\rho t}\, dtdn\Big] dkdk' \lesssim_q \|K\|_{L^q}$$

if $q \in [1, 2)$ and $\|A\|_{L^2(\mathbb{K}/\mathbb{M})} = \|B\|_{L^2(\mathbb{K}/\mathbb{M})} = 1$. The left-hand side is equal to

$$\int_{\mathbb{G}} K(g)\Big[\int_{\mathbb{K}} A(k)B(k(gk))e^{-\rho H(gk)}\, dk\Big]\, dg,$$

where

$$h = k(h)a_{H(h)}n(h)$$

is the decomposition of h in Iwasawa coordinates.

For $\lambda \in \mathcal{T} := \{\lambda \in \mathbb{C} : |\Im(\lambda)| < \rho\}$ set

$$I(\lambda) = \int_{\mathbb{G}} K(g)\Big[\int_{\mathbb{K}} A_\lambda(k)B_\lambda(k(gk))e^{(-\rho+i\lambda)H(gk)}\, dk\Big]\, dg,$$

$$A_\lambda(x) = A(x)|A(x)|^{i\lambda/\rho}, \qquad B_\lambda(x) = B(x)|B(x)|^{-i\lambda/\rho}.$$

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• Since

$$\|A_\lambda\|_{L^q} + \|B_\lambda\|_{L^{q'}} \leq 2, \qquad q = 2\rho/(\rho + \Re(i\lambda)),$$

we have

$$\sup_{\lambda \in \mathcal{T}} |I(\lambda)| \lesssim \|K\|_{L^1(\mathbb{G})}.$$

• Plancherel theorem (Harish-Chandra)

$$\Big[\int_{\mathbb{R}} \lambda^2 e^{-\lambda^2} \|\pi_\lambda(F)\|^2_{\mathrm{HS}}\, d\lambda \Big]^{1/2} \lesssim \|F\|_{L^2(\mathbb{G})},$$

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Lemma : If $F : \mathcal{T} \to \mathbb{C}$ is an analytic function and

$$\sup_{\lambda \in \mathcal{T}} |F(\lambda)| \leq N, \quad \int_{\mathbb{R}} |F(\lambda)| \lambda^2 e^{-\lambda^2}\, d\lambda \leq 1$$

then

$$|F(0)| \lesssim (\log(2+N))^3.$$

It follows that $I(0) \lesssim N^{1/2} (\log N)^{3/2}$ if K is the characteristic function of a set of measure $N \geq 2$.

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The convolution inequality

$$L^2(\mathbb{G}) * L^p(\mathbb{G}) \subseteq L^2(\mathbb{G}), \qquad p \in [1, 2).$$

• Kunze–Stein : on $SL(2, \mathbb{R})$ and the complex classical groups ;
• Lipsman (1969) : on $SO(d, 1)$;
• Eymard–Lohoué (1975) : on all connected semisimple Lie groups $\mathbb{G}$ with finite center, K-invariant kernels ;
• Cowling (1978) : on all connected semisimple Lie groups $\mathbb{G}$ with finite center.

• Cowling-Meda-Setti (1997), following earlier work of Lohoué-Rychener : on connected simple Lie groups $\mathbb{G}$ of real rank 1 with finite center

$$L^{p,a} * L^{p,b} \subseteq L^{p,c} \qquad \text{if } p \in (1, 2) \text{ and } 1/a + 1/b \geq 1 + 1/c.$$

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Weak Riesz rearrangement inequalities :

$$\int_{\mathbb{G}\times\mathbb{G}} u(g)v(g'g^{-1})w(g')\,dgdg' \leq C_{\mathbb{G}} \int_{\mathbb{G}\times\mathbb{G}} \widetilde{u}(g)\widetilde{v}(g'g^{-1})\widetilde{w}(g')\,dgdg',$$

where $\widetilde{u}, \widetilde{v}, \widetilde{w}$ are "symmetric rearrangements" of u, v, w.

This holds on :

• Euclidean spaces with $C_{\mathbb{R}^d} = 1$;

• Nilpotent Lie groups : if u, v, w are characteristic functions of sets U, V, W then

$$\int_{\mathbb{G}\times\mathbb{G}} u(g)v(g'g^{-1})w(g')\,dgdg' \leq \min(|U||V|, |U||W|, |V||W|)$$
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Theorem : (I. (2000)) The weak Riesz rearrangement inequality

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holds on connected simple Lie groups $\mathbb{G}$ of real rank 1 with finite center.

This is equivalent to the endpoint estimate

$$L^{2,1}(\mathbb{G}) * L^{2,1}(\mathbb{G}) \subseteq L^{2,\infty}(\mathbb{G}).$$

Open problem : does the weak Riesz rearrangement inequality hold on higher rank connected simple Lie groups $\mathbb{G}$ with finite center ?

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Open problem : does the weak Riesz rearrangement inequality hold on higher rank connected simple Lie groups $\mathbb{G}$ with finite center ?

Fourier analysis on symmetric spaces. The hyperbolic space $\mathbb{H}^d$ is defined as

$$\mathbb{H}^d = \{x \in \mathbb{R}^{d+1} : [x,x] = (x^0)^2 - (x^1)^2 - \dots (x^d)^2 = 1 \text{ and } x^0 > 0\},$$

with Riemannian metric induced by the Minkowski metric on $\mathbb{R}^{d+1}$. The hyperbolic space $\mathbb{H}^d$ can be identified with the homogeneous space $SO(d,1)/SO(d) = \mathbb{G}/\mathbb{K}$, and the group $SO(d,1)$ acts transitively on $\mathbb{H}^d$. Moreover, for any $g \in SO(d,1)$ the mapping $L_g : \mathbb{H}^d \to \mathbb{H}^d$, $L_g(x) = g \cdot x$, defines an isometry of $\mathbb{H}^d$.

The Cartan decomposition $\mathbb{G} = \mathbb{K}\mathbb{A}_+\mathbb{K}$: if $f \in C_0(\mathbb{H}^d)$ then

$$\int_{\mathbb{H}^d} f(x)\, d\mu = C \int_{\mathbb{K}} \int_{\mathbb{R}_+} f(ka_s \cdot \mathbf{0}) (\operatorname{sh} s)^{d-1}\, dkds,$$

where C is a suitable constant.

The Fourier–Helgason transform takes suitable functions defined on $\mathbb{H}^d$ to functions defined on $\mathbb{R} \times \mathbb{S}^{d-1}$. For $\omega \in \mathbb{S}^{d-1}$ (in the general settting of noncompact symmetric spaces, $\omega \in \mathbb{K}/\mathbb{M}$ where $\mathbb{M}$) and $\lambda \in \mathbb{C}$ (in general $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$), let $b(\omega) = (1, \omega) \in \mathbb{R}^{d+1}$ and

$$h_{\lambda,\omega} : \mathbb{H}^d \to \mathbb{C}, \quad h_{\lambda,\omega}(x) = [x, b(\omega)]^{i\lambda-\rho},$$

where $\rho = (d-1)/2$. It is known that

$$\Delta h_{\lambda,\omega} = -(\lambda^2 + \rho^2) h_{\lambda,\omega},$$

where Δ is the Laplace-Beltrami operator on $\mathbb{H}^d$. The Fourier transform of $f \in C_0(\mathbb{H}^d)$ is defined by the formula

$$\widetilde{f}(\lambda, \omega) = \int_{\mathbb{H}^d} f(x) h_{\lambda,\omega}(x)\, d\mu = \int_{\mathbb{H}^d} f(x)[x, b(\omega)]^{i\lambda-\rho}\, d\mu.$$

Fourier inversion formula : if $f \in C_0^\infty(\mathbb{H}^d)$ then

$$f(x) = \int_0^\infty \int_{\mathbb{S}^{d-1}} \widetilde{f}(\lambda,\omega)[x, b(\omega)]^{-i\lambda-\rho} |\mathbf{c}(\lambda)|^{-2} \, d\lambda d\omega,$$

where, for a suitable constant C,

$$\mathbf{c}(\lambda) = C \frac{\Gamma(i\lambda)}{\Gamma(\rho + i\lambda)}$$

is the Harish-Chandra $\mathbf{c}$-function on $\mathbb{H}^d$, and the invariant measure of $\mathbb{S}^{d-1}$ is normalized to 1.

Plancherel theorem : the Fourier transform $f \to \widetilde{f}$ extends to an isometry of $L^2(\mathbb{H}^d)$ onto $L^2(\mathbb{R}_+ \times \mathbb{S}^{d-1}, |\mathbf{c}(\lambda)|^{-2} d\lambda d\omega)$; moreover

$$\int_{\mathbb{H}^d} f_1(x)\overline{f_2(x)} \, d\mu = \frac{1}{2} \int_{\mathbb{R}\times\mathbb{S}^{d-1}} \widetilde{f_1}(\lambda,\omega)\overline{\widetilde{f_2}(\lambda,\omega)} |\mathbf{c}(\lambda)|^{-2} \, d\lambda d\omega,$$

for any $f_1, f_2 \in L^2(\mathbb{H}^d)$.

As a consequence, any bounded multiplier $m : \mathbb{R}_+ \to \mathbb{C}$ defines a bounded operator T_m on $L^2(\mathbb{H}^d)$ by the formula

$$\widetilde{T_m(f)}(\lambda, \omega) = m(\lambda) \cdot \widetilde{f}(\lambda, \omega).$$

Theorem : A necessary condition for boundedness on $L^p(\mathbb{H}^d)$ of the operator T_m is that the multiplier m extend to an even analytic function in the interior of the tube $\mathcal{T}_p = \{\lambda \in \mathbb{C} : |\Im \lambda| < |2/p - 1|\rho\}$.

Clerc–Stein (1974) in the setting of complex symmetric spaces.

Theorem : If $p \in (1,2)$ and m is an even analytic function in the tube $\mathcal{T}_p = \{\lambda \in \mathbb{C} : |\Im \lambda| < |2/p - 1|\rho\}$ that satisfies

$$\sup_{\lambda \in \mathcal{T}_p} (1 + |\lambda|)^j |\nabla^j m(\lambda)| \lesssim_j 1, \qquad j = 0, 1, \ldots,$$

then T_m defines a bounded operator on $L^p(\mathbb{H}^d)$.

- Clerc–Stein (1974) : complex symmetric spaces.
- Stanton–Tomas (1978) : symmetric spaces of real rank one.
- Anker–Lohoué (1986) : $\mathbb{G}$ is a normal real form.
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Idea of proof : the kernel of the operator T_m is

$$K(x) = c \int_{\mathbb{R}} m(\lambda)\Phi_\lambda(x)|\mathbf{c}(\lambda)|^{-2}\, d\lambda = K^0(x) + K^\infty(x),$$

where Φ_λ are the elementary spherical functions.
To bound the contribution of K^∞ we use the Herz majorizing principle :

$$\begin{aligned}\| * K^\infty\|_{L^p\to L^p} \lesssim & \sup_{\|A\|_{L^p(\mathbb{K}/\mathbb{M})}=\|B\|_{L^{p'}(\mathbb{K}/\mathbb{M})}=1} \int_{\mathbb{K}\times\mathbb{K}} A(k)B(k') \\ & \Big[\int_{\mathbb{R}\times\mathbb{N}} |K^\infty(k' a_t n k^{-1})| e^{2\rho t/p}\, dtdn\Big] dkdk' \\ & \lesssim_p \int_{\mathbb{R}_+} |K^\infty(a_t)| e^{-2\rho t/p'} (\operatorname{sh} t)^{d-1}\, dt.\end{aligned}$$

In special settings one can get good oscillatory bounds on $\Phi_\lambda(x)$, which give pointwise bounds on $|K^\infty|$. In the general setting, one can get L^2 bounds on K^∞ using the Abel transform (Taylor, Anker).

Singular integrals on symmetric spaces.

Theorem : I. (2002) Assume $p \in (1,2)$, $\rho_p = |2/p - 1|\rho$, and m is an even analytic function in the tube $\mathcal{T}_p = \{\lambda \in \mathbb{C} : |\Im\lambda| < \rho_p\}$ that satisfies the Hörmander–Michlin differential inequalities

$$|\nabla^j m(\lambda)| \lesssim_j \left(|\lambda - i\rho_p|^{-j} + |\lambda + i\rho_p|^{-j}\right), \qquad \lambda \in \mathcal{T}_p, j = 0, 1, \ldots.$$

Then T_m defines a bounded operator on $L^p(\mathbb{H}^d)$.

Main Proposition : Assume $p \in (1,2)$ and $\phi : \mathbb{R} \to \mathbb{C}$ is an even Calderon–Zygmund kernel, i.e.

$$(1+r)|\phi(r)| + (1+r^2)|\phi'(r)| \le 1, \quad r \in \mathbb{R},$$
$$\left|\int_0^N \phi(r)\,dr\right| \le 1, \quad N \in [0,\infty),$$

Let

$$K_p(k_1 a_t k_2) = e^{-2\rho t/p}\phi(t), \quad k_1, k_2 \in \mathbb{K}, t \in [0,\infty).$$

Then

$$\|*K_p\|_{L^p(\mathbb{H}^d)\to L^p(\mathbb{H}^d)} \lesssim_p 1.$$

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$$\| * K_p\|_{L^p(\mathbb{H}^d) \to L^p(\mathbb{H}^d)} \lesssim_p 1.$$

This type of singular integrals were extended to general symmetric spaces of the noncompact type, I. (2003) and Meda-Vallarino (2010).

Extensions to $L^p \to L^q$ estimates of operators defined by Fourier multipliers on noncompact symmetric spaces :

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